According to (3.14)

$$\cos\varphi = 2\cos^2(\varphi/2) - 1 \geqslant 8(1-\nu)(2-\nu)^{-2} - 1$$
$$(2-\nu)\cos\varphi + 3\nu\cos(\varphi_1+\varphi_2) \geqslant 8(1-\nu)/(2-\nu) - 2 + \nu - 3\nu =$$
$$2(\nu^2 - 5\nu + 2)/(2-\nu) > 0 \quad \text{for} \quad \nu < \nu_0$$

The proof of inequality (3.12) is completed.

By starting from the above exposition, an example can be constructed for $\forall\nu > \nu_0$ in which inequality (3.12) does not hold. Thus, an annular crack can be taken as $G_1(\varepsilon)$ and an annular crack with a tiny "expansion" (Fig.5) as $G_2(\varepsilon)$. Selection of the location of the point $\mathbf{R}(l)$ (in which (3.12) is not satisfied), and the "expansion", as well as the direction of the load $\mathbf{t}$ is shown in Fig.5, where the relationship $\sigma/2 = \operatorname{arc\,tg}\sqrt{1-\nu}$ should be satisfied.

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FINITE-PART INTEGRALS IN PROBLEMS OF THREE-DIMENSIONAL CRACKS*

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An effective method is proposed for solving the boundary integral equation (BIE) for the problem of a crack along a curvilinear surface in an elastic space on the basis of the transformation of the initial integrodifferential equation into an equation without derivatives. This is achieved by using the concept of the finite-part integral (FPI). Quadrature formulas are presented for such integrals over arbitrary convex polygons by approximating displacement discontinuities on the boundary by polynomials.

The well-known BIE for three-dimensional cracks contain either derivatives of the unknown functions or derivatives of a surface integral /1-7/. In both cases the presence of the derivatives significantly complicates the solution. However, as is shown in /8/, these difficulties are reduced in the case of a plane crack of normal discontinuity if the FPI concept is utilized /9, 10/. In this connection, it is useful to investigate the possibility of applying such an approach to the more general problem of a crack of arbitrary discontinuity and to develop the numerical side of its utilization. Both aims are pursued in this paper: the extension of this idea to the general case of three-dimensional cracks is given and methods are indicated for evaluating the integrals that originate by presenting quadrature formulas convenient for the numerical realization of the BIE method on a computer.

1. The consideration of the problem is based on the form of the BIE for three-dimensional cracks, which contains only derivatives of integrals over the surface but no derivatives of the displacement discontinuities under the integral sign /1, 6/. The integrals in the BIE have singularities generated by the term $1/r$ and combinations of its powers with differences between the coordinates of the control point x and the variable point of integration ξ (r is the distance between the points). This does not permit differentiation under the integral sign since it results in a non-integrable singularity (in the general case the original

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integral is already singular). The asymptotic behaviour of the terms under the integral sign in the displacement discontinuities does not depend on the curvature of the crack surface as $\xi \to x$ if this surface is sufficiently smooth, and is exactly the same as the behaviour of the corresponding terms in the problem of a plane crack of arbitrary discontinuity. Then eliminating and appending components corresponding to a certain piece of a plane crack touching the surface under consideration at the control point x. it is always possible to have singular terms just in the operator corresponding to the plane crack of arbitrary discontinuity. Consequently, it is sufficient to concentrate on just the latter case.

2. The BIE for a plane crack of arbitrary discontinuity in an infinite medium (/2/, for instance) in the form mentioned that contains only derivatives of intergrals, have the form

$$\sigma(\mathbf{x}) = kAw, \quad \tau_1(\mathbf{x}) = k\left[Au + \nu \frac{\partial}{\partial x_2}(Bu - Cv)\right] \tag{2.1}$$

$$\tau_2(\mathbf{x}) = k\left[Av - \nu \frac{\partial}{\partial x_1}(Bu - Cv)\right], \quad \mathbf{x} \in S; \quad k = \frac{E}{8\pi(1-\nu^2)}$$

$$Aw = \Delta \int_S \frac{w(\xi)}{r}\, dS, \quad Bu = \int_S \frac{x_2 - \xi_2}{r^3} u(\xi)\, dS$$

$$Cv = \int_S \frac{x_1 - \xi_1}{r^3} v(\xi)\, dS$$

Here $\sigma(\mathbf{x})$, $\tau_1(\mathbf{x})$, $\tau_2(\mathbf{x})$ are components of the stress vector at the point $\mathbf{x}$ of the crack surface S along the x_3, x_1, x_2 axes (the x_1, x_2 axes are in the plane of the crack while the x_3 axis is perpendicular), $w(\xi)$, $u(\xi)$, and $v(\xi)$ are components of the displacement discontinuity along the x_3, x_1, x_2 axes at the point of integration ξ of the surface S, where the discontinuities are evaluated as the difference between the displacements of the crack upper and lower edges if the lower is considered the edge for which the x_3 axis is the external normal with respect to the domain it bounds; $\Delta = \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2$ is the Laplace operator, E, ν are Young's modulus and Poisson's ratio of the medium, and the integrals marked with a bar are understood in the principal-value sense.

It is proved /8/ for the operator A corresponding to the case of a crack of normal discontinuity that the formal transfer of the Laplace operator under the integral sign to yield $\Delta(1/r) = 1/r^3$ becomes legitimate if the divergent integral obtained, that does not exist even in the principal value sense, i.e.

$$Aw = \text{v.f.} \int_S \frac{w(\xi)}{r^3}\, dS, \quad \mathbf{x} \in S \tag{2.2}$$

is treated in a special manner.

This formula acquires meaning if the integral on the right-hand side, marked with the symbols v.f., is considered as a FPI. In the case being studied the order of the singularity only exceeds the dimensionality of the domain of integration by one, and consequently, the FPI, in conformity with its definition /9, 10/, can be evaluated according to the following rule /8/. A circular domain S_0 of arbitrary radius r_0 is separated out around the point $\mathbf{x}$, a term $w(\mathbf{x})/r^3$ is subtracted and added and the integral of $1/r^3$ is evaluated formally by substituting $r = r_0$. By definition

$$\Delta \int_{S_0} \frac{1}{r}\, dS = \text{v.f.} \int_{S_0} \frac{1}{r^3} r\, dr\, d\varphi = -\frac{2\pi}{r_0}$$

Attention can here be turned to the fact that the FPI of a positive function is a negative number. If the function w has an analytic expression in the domain S_0 and the integral of w/r^3 is taken in quadratures, then the FPI can be evaluated by substituting the integration limits corresponding to the boundary S_0 in conformity with the formal expression.

It is useful to perform an analogous passage to differentiation under the integral sign in the formulas or the tangential force vector components τ_1, τ_2 also. In addition to (2.2) we hence obtain

$$\frac{\partial}{\partial x_1} Bu = -3\text{v.f.} \int_S \frac{(x_1 - \xi_1)(x_2 - \xi_2)}{r^5} u(\xi)\, dS \tag{2.3}$$

$$\frac{\partial}{\partial x_1} Cv = \text{v.f.} \int_S \frac{1}{r^3} v(\xi)\, dS - 3\text{v.f.} \int_S \frac{(x_1 - \xi_1)^2}{r^5} v(\xi)\, dS$$

$$\frac{\partial}{\partial x_2} Bu = \text{v.f.} \int_S \frac{1}{r^3} u(\xi)\, dS - 3\text{v.f.} \int_S \frac{(x_2 - \xi_2)^2}{r^5} u(\xi)\, dS$$

$$\frac{\partial}{\partial x_2} Cv = -3\text{v.f.} \int_S \frac{(x_1 - \xi_1)(x_2 - \xi_2)}{r^5} v(\xi)\, dS$$

The proof is presented in the example of the second of these formulas. We have

$$\frac{\partial}{\partial x_1} Cv = \frac{\partial}{\partial x_1} \int_S \frac{x_1 - \xi_1}{r^3} v(\xi)\, dS = \frac{\partial}{\partial x_1} \int_{S/K} \frac{x_1 - \xi_1}{r^3} v(\xi)\, dS +$$

$$\frac{\partial}{\partial x_1} \int_K \frac{x_1 - \xi_1}{r^3} [v(\xi) - v(\mathbf{x})]\, dS + \frac{\partial}{\partial x_1} \left[v(\mathbf{x}) \int_K \frac{x_1 - \xi_1}{r^3}\, dS \right]$$

where K is a certain neighbourhood of the point $\mathbf{x}$ belonging to S, and S/K is the complement to S. The first two components can be differentiated under the integral sign. Then

$$\frac{\partial}{\partial x_1} Cv = \int_{S/K} \left[\frac{1}{r^3} - 3\frac{(x_1 - \xi_1)^2}{r^5} \right] v(\xi) dS + \int_K \left\{ \left[\frac{1}{r^3} - 3\frac{(x_1 - \xi_1)^2}{r^5} \right] [v(\xi) -\right.$$

$$\left. v(\mathbf{x})] - \frac{\partial v(\mathbf{x})}{\partial x_1} \frac{x_1 - \xi_1}{r^3} \right\} dS + \frac{\partial v(\mathbf{x})}{\partial x_1} \int_K \frac{x_1 - \xi_1}{r^3} dS + v(\mathbf{x}) \frac{\partial}{\partial x_1} \int_K \frac{x_1 - \xi_1}{r^3} dS$$

The second integral on the right-hand side of this equation is represented in the form

$$\text{v.f.} \int_K \frac{1}{r^3} v(\xi)\, dS - v(\mathbf{x})\, \text{v.f.} \int_K \frac{1}{r^3}\, dS - 3\text{v.f.} \int_K \frac{(x_1 - \xi_1)^2}{r^5} v(\xi)\, dS +$$

$$3v(\mathbf{x})\, \text{v.f.} \int_K \frac{(x_1 - \xi_1)^2}{r^5}\, dS - \frac{\partial v(x)}{\partial x_1} \int_K \frac{x_1 - \xi_1}{r^3}\, dS$$

from which we obtain an equation that differs from the second equation in (2.3) by the presence of the component $v(\mathbf{x}) J$ on the right-hand side, where

$$J = \frac{\partial}{\partial x_1} \int_K \frac{x_1 - \xi_1}{r^3} dS - \text{v.f.} \int_K \frac{1}{r^3} dS + 3\, \text{v.f.} \int_K \frac{(x_1 - \xi_1)^2}{r^5} dS$$

It remains to prove that $J = 0$. A circular domain S_0 is used as the domain K in /8/ in the proof of this equation. In order to illustrate the independence of the result from the selection of the domain K here, a rectangle $a \leqslant \xi_1 \leqslant b$; $c \leqslant \xi_2 \leqslant d$ is taken. The point $\mathbf{x}$ lies within the rectangle. Integration in quadratures yields

$$\frac{\partial}{\partial x_1} \int_K \frac{x_1 - \xi_1}{r^3} dS = \frac{1}{b_1} \left(\frac{b_2}{b_3} - \frac{a_2}{b_4} \right) + \ldots \tag{2.4}$$

$$a_1 = x_1 - a, \quad a_2 = x_2 - c, \quad b_1 = x_1 - b, \quad b_2 = x_2 - d$$

$$a_3^2 = a_1^2 + a_2^2, \quad a_4^2 = a_1^2 + b_2^2, \quad b_3^2 = b_1^2 + b_2^2, \quad b_4^2 = a_2^2 + b_1^2$$

(the three dots denotes the component obtained from the previous commutation of a and b). The second and third integrals in the expression for J, which neither exist in the ordinary nor the principal value sense, are evaluated formally in conformity with the definition, by substituting the limits of integration corresponding to the boundary of the domain K into the analytic formulas obtained. Consequently, we obtain for the rectangular domain K

$$\text{v.f.} \int_K \frac{1}{r^3} dS = \frac{1}{a_2} \left(\frac{b_4}{b_1} - \frac{a_3}{a_1} \right) + \ldots \tag{2.5}$$

$$\text{v.f.} \int_K \frac{(x_1 - \xi_1)^2}{r^5} dS = \frac{1}{3a_1} \left(\frac{a_4}{b_2} - \frac{a_3}{a_2} + \frac{b_2}{a_4} - \frac{a_2}{a_3} \right) + \ldots$$

where it can be verified that the first of Eqs.(2.5) agrees with the result of the calculation

$$\Delta \int_K \frac{1}{r} dS$$

It follows from (2.4) and (2.5) that $J = 0$, i.e., the validity of the second equation in (2.3).

The BIE system (2.1) is written in the following final form by using (2.2) and (2.3)

$$\sigma(\mathbf{x}) = k\, \text{v.f.} \int_S \frac{w(\xi)}{r^3} dS \tag{2.6}$$

$$\tau_1(\mathbf{x}) = k \left[(1 + \nu)\, \text{v.f.} \int_S \frac{1}{r^3} u(\xi)\, dS - 3\nu\, \text{v.f.} \int_S \frac{(x_2 - \xi_2)^2}{r^5} u(\xi)\, dS + \right.$$

$$\left. 3\nu\, \text{v.f.} \int_S \frac{(x_1 - \xi_1)(x_2 - \xi_2)}{r^5} v(\xi)\, dS \right],$$

$$\tau_2(x) = k\left[(1+\nu)\,\text{v.f.}\int\limits_S \frac{1}{r^3} v(\xi)\,dS - 3\nu\,\text{v.f.}\int\limits_S \frac{(x_1-\xi_1)^2}{r^5} v(\xi)\,dS\right.$$
$$\left.3\nu\,\text{v.f.}\int\limits_S \frac{(x_1-\xi_1)(x_2-\xi_2)}{r^5} u(\xi)\,dS\right]$$

which is convenient for applications in that the formulas contain no derivatives.

It is evident that since the circular domain S_0 can be enclosed in a sufficiently arbitrary domain of any configuration K and the integral over K/S_0 has no singularities, any other domain enclosing a circle, particularly triangles or rectangles, can be taken as the subdomain in which the integral is formally evaluated. For such a domain, the integral of a function with a non-integrable singularity can also be taken formally and appropriate limits of integration corresponding to the domain boundary can be substituted. In addition to the main idea of going over from (2.1) to (2.6) the integration over the rectangle used above illustrated, this fact, which is important for all its obviousness, since it permits a substantial simplification of the calculations by using the same elementary cells (triangles, squares, and rectangles) as the domains K over which the FPI are calculated into which the surface S is separated in the discretization of the problem. This advantage was utilized (and substantially increased the efficiency of the calculation as compared with the selection of the circular domain S_0 as K) in /11/*, (*See also: Zubkova I.A., Development of a method of analysing elevated mountain pressure zones on the basis of solving a three-dimensional problem on the stress distribution around cleaning drifts. Candidate Dissertation, National Scientific-Research Surveying Institute, Leningrad, 1983.) where the integration was performed over squares.

3. It is useful to take one more step in developing methods of evaluating the divergent integrals being obtained: not only to use the arbitrary domains of integration K surrounding the control point $\mathbf{x}$ but also to get rid of the condition that the integral over such a domain would necessarily exist as a formal analytic expression. This can be achieved by considering the FPI as the limits of the corresponding ordinary integrals. The formulas obtained are a natural corollary of the fact that the BIE themselves are the result of an analogous passage to the limit from the body under consideration onto the surface S.

The following equality can be obtained from (2.2) for the density $f(\xi)$ having continuous derivatives to second order inclusive in the neighbourhood of the point $\xi = \mathbf{x}$:

$$I_1 = \text{v.f.}\int\limits_K \frac{1}{r^3} f(\xi)\,dS = \Delta\int\limits_K \frac{1}{r} f(\xi)\,dS = \lim_{x_3\to 0}\left[\int\limits_K \frac{1}{R^3} f(\xi)\,dS - \right.$$
$$\left.\frac{2\pi}{x_3} f(\mathbf{x})\right],\quad R = [(x_1-\xi_1)^2 + (x_2-\xi_2)^2 + x_3^2]^{1/2}$$

The properties of a double-layer potential are used here.

Calculations for the righ-hand side can be performed in a cylindrical system of coordinates with origin at the point $\mathbf{x}$ and coordinates ρ, φ, z, related to the original Cartesian coordinates by means of the formulas $\xi_1 - x_1 = \rho\cos\varphi$, $\xi_2 - x_2 = \rho\sin\varphi$, $x_3 = z$. In evaluating the integral in the right side, the term annihilating the component $2\pi f(\mathbf{x})/x_3$ is separated out at once. Moreover, since $f(\xi)$ possesses continuous derivatives to second order inclusive, the following representation holds:

$$f(\xi) - f(\mathbf{x}) = \rho g_1(\mathbf{x}, \rho, \varphi) + \rho^2 g_2(\mathbf{x}, \rho, \varphi)$$

where $g_1 = \cos\varphi\,\partial f/\partial x_1 + \sin\varphi\,\partial f/\partial x_2$, $g_2(\mathbf{x}, \rho, \varphi)$ is a continuous function and $\lim g_2(\mathbf{x}, \rho, \varphi)\rho^2 = 0$ as $\rho \to 0$.

Then by integration and passage to the limit, we obtain

$$I_1 = \int\limits_0^{2\pi} g_1 \ln\rho(\varphi)\,d\varphi - f(\mathbf{x})\int\limits_0^{2\pi}\frac{d\varphi}{\rho(\varphi)} + \int\limits_0^{2\pi} d\varphi \int\limits_0^{\rho(\varphi)} g_2\,d\rho$$

where $\rho(\varphi)$ is understood to be the value of ρ at a point of the contour of the domain K that has the angular coordinate φ.

This deduction is essentially the reproduction of calculations in the proof of the continuity of the double-layer potential, while the right-hand side is a well-known expression /12, p.239/.

By using potential theory, expressions can also be obtained in an anlogous manner for the three other kinds of integrals in (2.6)

$$I_2 = \text{v.f.}\int\limits_K \frac{(x_1-\xi_1)^2}{r^5} f(\xi)\,dS = \int\limits_0^{2\pi} g_1 \cos^2\varphi\,\ln\rho(\varphi)\,d\varphi - \qquad (3.1)$$

$$f(x)\int_0^{2\pi}\frac{\cos^2\varphi}{\rho(\varphi)}d\varphi+\int_0^{2\pi}\cos^2\varphi\,d\varphi\int_0^{\rho(\varphi)}g_2\,d\rho$$

$$I_3=\text{v.f.}\int_K\frac{(x_2-\xi_2)^2}{r^3}f(\xi)\,dS=\int_0^{2\pi}g_1\sin^2\varphi\ln\rho(\varphi)\,d\varphi-$$

$$f(x)\int_0^{2\pi}\frac{\sin^2\varphi}{\rho(\varphi)}d\varphi+\int_0^{2\pi}\sin^2\varphi\,d\varphi\int_0^{\rho(\varphi)}g_2\,d\rho,\quad I_4=$$

$$\text{v.f.}\int_K\frac{(x_1-\xi_1)(x_2-\xi_2)}{r^3}f(\xi)\,dS=\int_0^{2\pi}g_1\sin\varphi\cos\varphi\ln\rho(\varphi)\,d\varphi-$$

$$f(x)\int_0^{2\pi}\frac{\sin\varphi\cos\varphi}{\rho(\varphi)}d\varphi+\int_0^{2\pi}\sin\varphi\cos\varphi\,d\varphi\int_0^{\rho(\varphi)}g_2\,d\rho$$

We have $g_1=g_2=0$ for the constant of the function $f(\xi)$ in the domain K. There hence follows for the triangular element (figure, a) for $f(\xi)=1$

$$I_1=-\sum_{i=1}^{3}\frac{1}{d_i}(\sin\psi_i+\sin\chi_i)$$

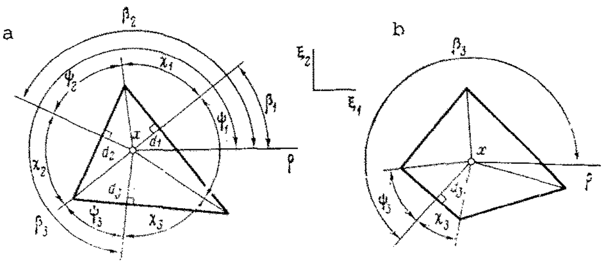
$$I_2=-\sum_{i=1}^{3}\frac{1}{d_i}\Big\{\cos\beta_i\Big[\sin(\beta_i+\chi_i)-\sin(\beta_i-\psi_i)-$$
$$\frac{\sin^3(\beta_i+\chi_i)-\sin^3(\beta_i-\psi_i)}{3}\Big]-\sin\beta_i\frac{\cos^3(\beta_i+\chi_i)-\cos^3(\beta_i-\psi_i)}{3}\Big\}$$

$$I_3=I_1-I_2,\quad I_4=-\sum_{i=1}^{3}\frac{1}{d_i}\Big[\cos\beta_i\frac{\cos^3(\beta_i-\psi_i)-\cos^3(\beta_i+\chi_i)}{3}-$$
$$\sin\beta_i\frac{\sin^3(\beta_i-\psi_i)-\sin^3(\beta_i+\chi_i)}{3}\Big]$$

The meaning of the notation $d_i, \psi_i, \chi_i, \beta_i$ is clear from Fig.1a.

In the case of a right triangle with side a and point x at the centre, we have $I_1=-18/a$; $I_2=I_3=-9/a$; $I_4=0$ when one of the vertices is on the x_1 or x_2 axis.

For the quadrangular domain (figure, b) the summation is extended up to four in the formulas written for the triangular element and the meaning of the notation is as before. Formulas for an arbitrary convex polygon are obtained analogously.



In the case of a rectangle with sides parallel to the coordinate axes, we have expressions for I_1 and I_2 that have already been obtained in Sect.2 by another method, while the obtain for I_3 and I_4

$$I_3=I_1-I_2,\quad I_4=\frac{1}{3}\left(\frac{1}{a_3}+\frac{1}{b_3}-\frac{1}{a_4}-\frac{1}{b_4}\right)$$

If the point x is at the intersection of the diagonals of a rectangle with sides a, b, then

$$I_1=-8\frac{\sqrt{a^2+b^2}}{ab},\quad I_2=-\frac{16}{3}\frac{a^4+b^4+3a^2b^2}{ab\,(a^2+b^2)^{3/2}},\quad I_3=I_1-I_2,\quad I_4=0$$

If the density has the form $f(x)=c_0+c_1x_1+c_2x_2+c_3x_1x_2$, then $g_1(x)=(c_1+c_3x_2)\cos\varphi+(c_2+c_3x_1)\sin\varphi$, $g_2(x)=c_3\sin\varphi\cos\varphi$ and formulas (3.1) for the FPI yield

$$I_1=(c_1+c_3x_2)\int_0^{2\pi}\cos\varphi\ln\rho(\varphi)\,d\varphi+(c_2+c_3x_1)\int_0^{2\pi}\sin\varphi\ln\rho(\varphi)\,d\varphi+$$

$$c_3\int_0^{2\pi}\rho(\varphi)\sin\varphi\cos\varphi\,d\varphi-(c_0+c_1x_1+c_2x_2+c_3x_1x_2)\int_0^{2\pi}\frac{d\varphi}{\rho(\varphi)}$$

$$I_2=(c_1+c_3x_2)\int_0^{2\pi}\cos^3\varphi\ln\rho(\varphi)\,d\varphi+(c_2+c_3x_1)\times$$

$$\int_0^{2\pi}\sin\varphi\cos^2\varphi\ln\rho(\varphi)\,d\varphi+c_3\int_0^{2\pi}\rho(\varphi)\sin\varphi\cos^3\varphi\,d\varphi-$$

$$(c_0+c_1x_1+c_2x_2+c_3x_1x_2)\int_0^{2\pi}\frac{\cos^2\varphi}{\rho(\varphi)}\,d\varphi,$$

$$I_3=I_1-I_2,\quad I_4=(c_1+c_3x_2)\int_0^{2\pi}\sin\varphi\cos^2\varphi\ln\rho(\varphi)\,d\varphi+(c_2+$$

$$c_3x_1)\int_0^{2\pi}\sin^2\varphi\cos\varphi\ln\rho(\varphi)\,d\varphi+c_3\int_0^{2\pi}\rho(\varphi)\sin^2\varphi\cos^2\varphi\,d\varphi-$$

$$(c_0+c_1x_1+c_2x_2+c_3x_1x_2)\int_0^{2\pi}\frac{\sin\varphi\cos\varphi}{\rho(\varphi)}\,d\varphi$$

For the numerical solution of problems, the coefficients c_0, c_1, c_2, c_3 are determined from the condition that the approximating function takes specific values at the four nodal points of the cell. Formulas for expressing $f(x)$ by higher-order polynomials are also obtained analogously. Other approximations can also be utilized. Numerical integration is hence used in cases when the integrals in (3.1) are not expressed by analytic formulas.

The results presented yield a convenient means for solving BIE for cracks of arbitrary discontinuity since the passage to the form (2.6) eliminates differentiation, while utilization of the formulas obtained makes evaluation of the integrals (3.1) no more complicated than the evaluation of ordinary integrals. By separating the crack surface into elementary cells and giving the approximation of the desired function (or known function, in mixed problems), integration over the cell containing it can be performed for each control point using the formulas presented. For the remaining cells the integrals have no singularities and are taken by ordinary methods. The remaining calculations are also completely traditional.

In conclusion, we note that the concept of the FPI introduced by Hadamard /13/ and contributing to the creation of the thoery of generalized functions, has now been interpreted within the framework of this theory. Both formal questions of regularizing integrals of the type under consideration, as well as questions associated with their evaluation /14/ are solved in it.

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THE ASYMPTOTIC STABILITY OF SYSTEMS WITH DELAY*

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Sufficient conditions for the existence of a finite domain of attraction of an unperturbed solution of autonomous system with delay are obtained, and its lower estimate is given using a method which requires that only the Lyapunov function need be known for the system in question without delay.

The results of investigating the stability of non-linear systems with delay /1, 2/ enable one to determine the domain of stability in the parameter space of the non-linear problem and of the domain of attraction of the unperturbed solution for mainly autonomous and non-autonomous first-order systems.

Following /3/, the application of the Lyapunov vector function is proposed in order to use the methods described in /1, 2/ constructively for systems of higher order.

Let the unperturbed motion $x=0$ of the system

$$x_i^{\cdot} = f_i(x(t)) + \sum_{j=1}^{m} F_{ij}(x(t))\, u_j(x(t-\tau)), \quad \tau = \mathrm{const} \geqslant 0 \tag{1}$$

$$x \in R^n, \quad u \in R^m, \quad f_i,\ F_{ij},\ u_j \in C^1(\Omega), \quad \Omega \subset R^n, \quad m \leqslant n$$

without delay $(\tau=0)$ by asymptotically stable, and let a Lyapunov function $V(x)$, positive definite in the convex region $\Omega_0 \subset \Omega$ be known for (1), with the time dertivative of this function negative definite in Ω by virtue of the system (1) $(\tau=0)$. Some or all (when $m=n$) functions $f_i(x(t))$ here can be identically equal to zero.

We take $t=0$ as the initial instant. Let the initial continuous curve be described, at $\tau \leqslant t \leqslant 0$, by the function $\Phi(t) \in \Omega$. We shall write it in the form of a sum $\Phi(t) = \varphi(t) + \psi(t)$ where $\|\varphi(t)\| \leqslant \varphi^*$, $\varphi^* = \mathrm{const}$ ($\|\cdot\|$ is the Euclidean norm of a vector) and the function $\psi(t)$, $\psi(0) = x(0)$ is a solution of system (1) for $\tau=0$. We will assume, without loss of generality, that $\varphi(0)=0$ and call the function $\psi(t)$ the reference function.

Let the domain Ω^* together with its boundary $\partial\Omega^*$ defined by the equation $V(x) = v^*$, $v^* = \mathrm{const} > 0$ lie within Ω_0. Such a domain will be the domain of attraction of the unperturbed solution $x=0$ of system (1) at $\tau=0$ /1/. The domain may collapse when $\tau \neq 0$. Below we shall consider the bounded domains only. If on the other hand the motion $x=0$ is asymptotically stable in the large when $\tau=0$, then the bounded domain Ω_0 can be chosen arbitrarily.

We shall regard as the domain of attraction of the unperturbed motion $x=0$ of system (1), the sets of points of the phase space representing the initial values of the solutions of (1) tending, as $t\to\infty$, to the unperturbed motion $x=0$ for any initial functions $\Phi(t)$ belonging to the class specified above.

Let us clarify the conditions imposed on the parameters τ and φ^* under which the region Ω^* remains within the domain of attraction of the unperturbed motion.

Let us write the solution of system (1) $(\tau\neq 0)$ within the time interval $\tau \leqslant t \leqslant 0$ in the form of a sum $x(t) = \psi(t) + y(t)$ where $y(t) = \varphi(t)$ and $t \in [-\tau, 0]$. Then, when $t \geqslant 0$, the function $y(t)$ will, according to (1), satisfy the differential vector equation which can be written in the form

$$y^{\cdot} = \sum_{k=1}^{4} F_k(y, \psi, \tau) \tag{2}$$

$$F_1 = f(\psi+y) + F(\psi+y)\, u(\psi+y) - f(\psi) - F(\psi)\, u(\psi)$$

$$F_2 = F(\psi+y)\, [u(\psi) - u(\psi+y)], \quad F_3 = F(\psi+y)\, \{u[\psi(t-\tau) + y(t-\tau)] - u[\psi(t-\tau)]\}, \quad F_4 = F(\psi+y)\, \{u[\psi(t-\tau)] - u[\psi(t)]\}$$

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